A Nekrasov-Okounkov formula in type $\tilde{C}$

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Plan

1. Partitions and Macdonald's formula
2. A new bijection φ between vectors and partitions
3. A Nekrasov-Okounkov formula in type $\tilde{C}$
4. A generalization through Littlewood decomposition

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Partitions

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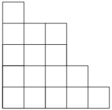

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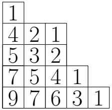

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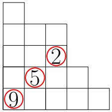

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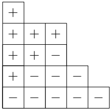

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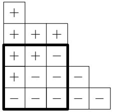

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$\delta_\lambda = \begin{cases} +1 & \text{if the Durfee square of } \lambda \text{ is even} \\ -1 & \text{else} \end{cases}$

$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of t

t-cores

Let $t \geq 2$ be an integer. A partition is a *t-core* if its hook lengths set does not contain t. It is equivalent to the fact that the hook lengths set does not contain any integral multiple of t, , $i.e.$ $\mathcal{H}_t(\lambda) = \emptyset$.

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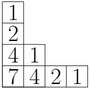

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Han (2009): expansion of η function in terms of hooks

Dedekind η function

We define Dedekind eta function by $\eta(x) = x^{1/24} \prod_{i \geq 1} (1 - x^i)$.

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Lehmer's conjecture (1947)

Coefficents of expansion of η^{24} are nonzero.

Macdonald formula in type $\tilde{A}$

Theorem (Macdonald, 1972)

For any odd integer t, we have:

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0,v_1,\ldots,v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2+v_1^2+\cdots+v_{t-1}^2)/2t}, \qquad (1)$$

where the sum is over t-tuples of integers $(v_0, \ldots, v_{t-1}) \in \mathbb{Z}^t$ *such that* $v_0 + \cdots v_{t-1} = 0$ *and* $v_i \equiv i \bmod t$.

Nekrasov-Okounkov formula in type $\tilde{A}$

Theorem (Nekrasov-Okounkov, 2003; Han, 2009)

For any complex number z we have

$$\prod_{k\geq 1}(1-x^k)^{z-1} = \sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right).$$

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Macdonald in type $\tilde{C}$

Theorem (Macdonald, 1972)

For any integer $t \geq 2$, we have:

$$\eta(X)^{2t^2+t} = c_1 \sum \prod_i v_i \prod_{i<j} ({v_i}^2 - v_j^2) X^{\|v\|^2/4(t+1)},$$

where the sum ranges over $(v_1, \ldots, v_t) \in \mathbb{Z}^t$ such that $v_i \equiv i \mod 2t+2$.

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We write $v_i = (2t+2)n_i + i$.

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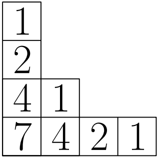

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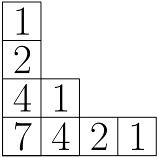

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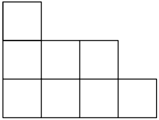

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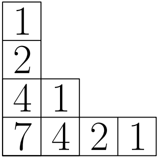

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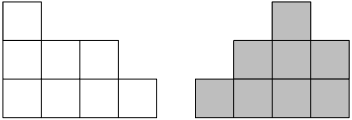

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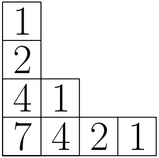

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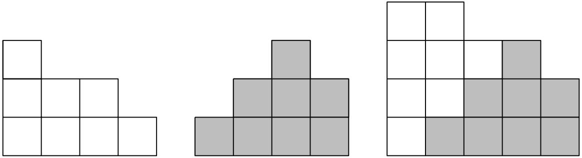

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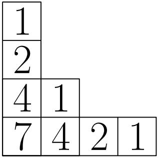

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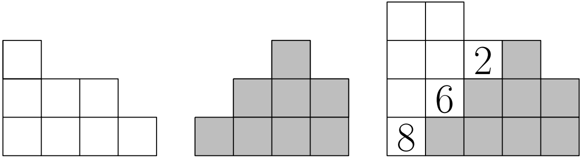

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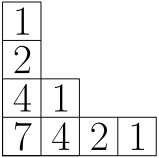

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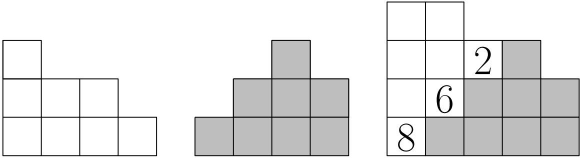

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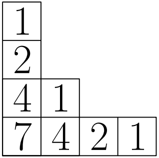

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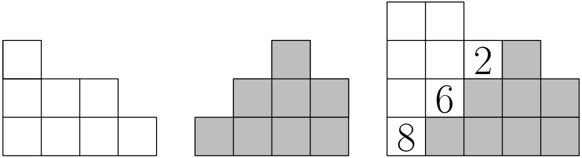

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Theorem (P., 2014)

The generating function for pairs of self-conjugate and doubled distinct t-cores is:

$$\sum_{(\lambda,\mu)\in SC_{(t)}\times DD_{(t)}} q^{|\lambda|+|\mu|} = \frac{(q^2;q^2)_\infty (q^t;q^t)_\infty (q^{2t-1};q^{2t-1})_\infty^{t-2}}{(q;q)_\infty}$$

Some properties

Let λ be a self-conjugate (resp. doubled distinct) (t+1)-core, and h be one of its principal hook length.

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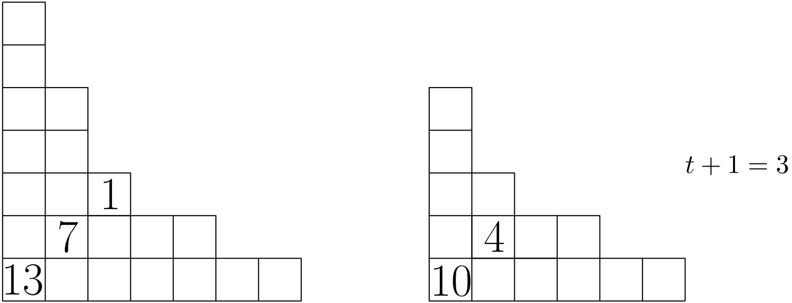

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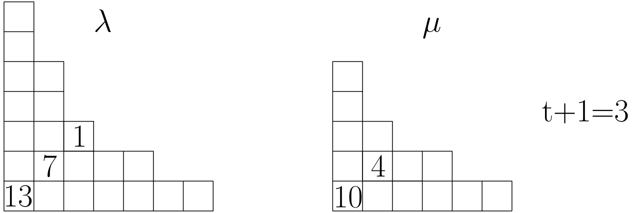

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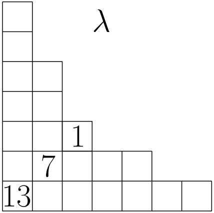
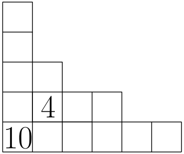


μ



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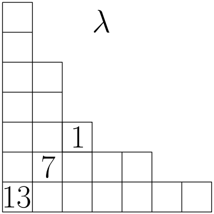

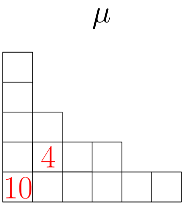

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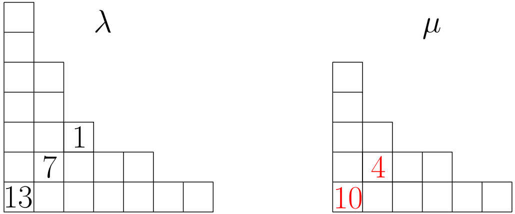

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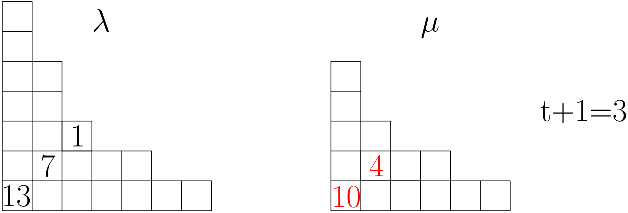

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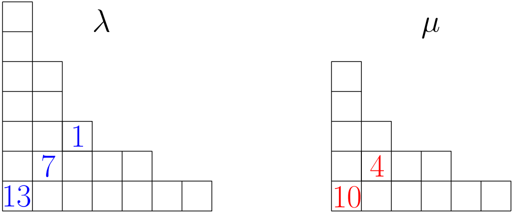

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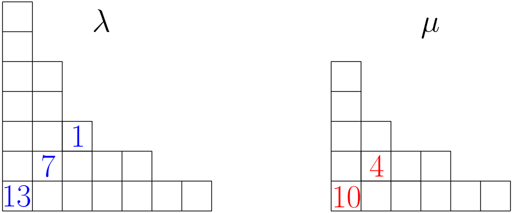

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$\Rightarrow n_2 = \frac{-(3+\Delta_2)-2}{6} = -3$

Definition of bijection φ

Theorem (P., 2014)

Let t an integer ≥ 2. *The map*

$$\begin{array}{rccc} \varphi : & SC_{(t+1)} \times DD_{(t+1)} & \to & \mathbb{Z}^t \\ & (\lambda, \mu) & \mapsto & (n_1, \ldots, n_t) \end{array}$$

is a bijection such that:

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$$\begin{array}{cccc} \varphi : & SC_{(t+1)} \times DD_{(t+1)} & \to & \mathbb{Z}^t \\ & (\lambda, \mu) & \mapsto & (n_1, \ldots, n_t) \end{array}$$

is a bijection such that:

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Definition of bijection φ

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- $\prod_i [(2t+2)n_i + i] \prod_{i<j} \left[((2t+2)n_i + i)^2 - ((2t+2)n_j + j)^2\right] =$

$$\frac{\delta_\lambda \delta_\mu}{c_1} \prod_{h_{ii}} \left(1 - \frac{2t+2}{h_{ii}}\right)\left(1 - \frac{t+1}{h_{ii}}\right) \prod_{j=1}^{h_{ii}-1} \left(1 - \left(\frac{2t+2}{h_{ii} + \tau_j j}\right)^2\right)$$

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A Nekrasov-Okounkov formula in type $\tilde{C}$

Theorem (P., 2014)

For any complex number z we have

$$\prod_{k\geq 1}(1-x^k)^{2z^2+z} = \sum_{(\lambda,\mu)\in \mathcal{S}C\times DD} \delta_\lambda \delta_\mu x^{|\lambda|+|\mu|}$$

$$\times \prod_{h_{ii}} \left(1-\frac{2z+2}{h_{ii}}\right)\left(1-\frac{z+1}{h_{ii}}\right)\prod_{j=1}^{h_{ii}-1}\left(1-\left(\frac{2z+2}{h_{ii}+\tau_j j}\right)^2\right)$$

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- Apply bijection φ to obtain the previous formula for any integer $t \geq 2$, except that the sum ranges over (t+1)-cores
- Replace the previous sum by a sum over all partitions in $SC \times DD$
- Check that coefficents of x^n on both sides are polynomials in t, and conclude that the formula is true for any complexe number z

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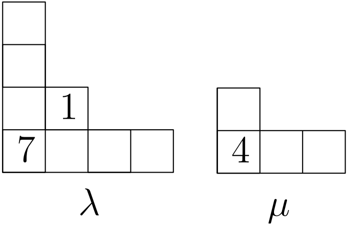
- $$\prod_{h_{ii} \in \Delta} \left(1 - \frac{2t+2}{h_{ii}}\right)\left(1 - \frac{t+1}{h_{ii}}\right) \prod_{j=1}^{h_{ii}-1} \left(1 - \left(\frac{2t+2}{h_{ii} + \tau_j j}\right)^2\right) = \prod_{h \in \nu} \left(1 - \frac{2t+2}{h\,\varepsilon_h}\right),$$

where ε_h is equal to -1 is the box h is strictly above the principal diagonal, and to 1 otherwise.

The partition ν



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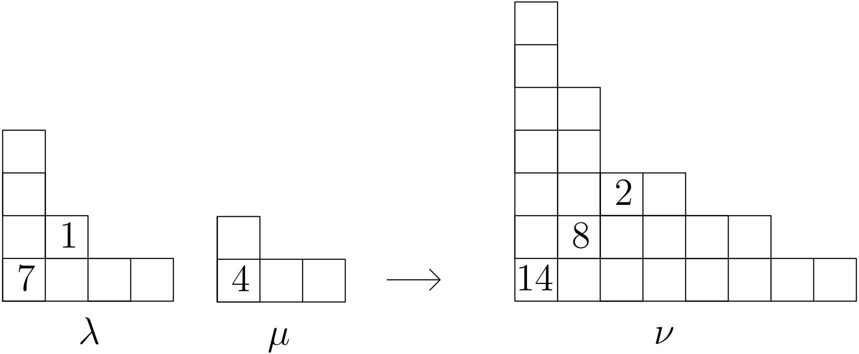

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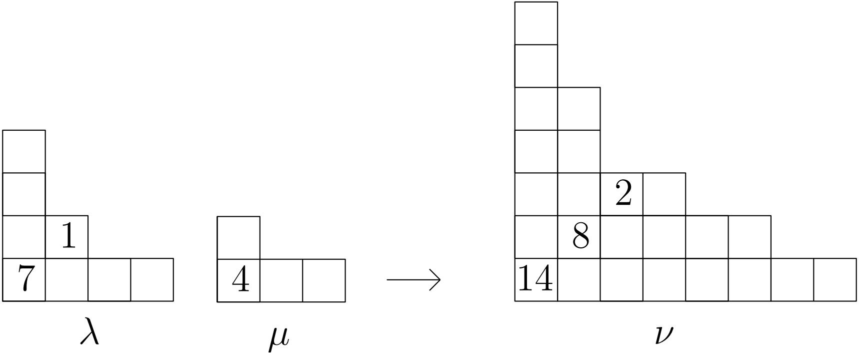

We verify: $7 + 1 + 4 = 12 = (14 + 8 + 2)/2$.

Main Theorem

Theorem (P., 2014)

For any complex number t, the following expansion holds:

$$\prod_{k\geq 1}(1-x^k)^{2z^2+z} = \sum_{\lambda\in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{2z+2}{h\,\varepsilon_h}\right),$$

where the sum ranges over doubled distinct partitions, δ_λ is equal to 1 (resp. -1) if the Durfee sqare of λ is of even (resp. odd) size, and ε_h is -1 if the box h is strictly above the diagonal and 1 otherwise.

Some applications

- Taking $z = -1$ yields this famous expansion,

$$\prod_{n\geq 1}(1 - x^n) = \sum_{\lambda}(-1)^{\#\{\text{parts of } \lambda\}} x^{|\lambda|},$$

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- For any positive integer n,

$$\sum_{\substack{\lambda\in DD \\ |\lambda|=2n}} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h} = \frac{1}{2^n n!}.$$

This is a symplectic analogous of the hook formula.

Extensions to other types

Theorem

The previous Theorem generalized the following families of formula:

(i) *Macdonald's formula in type* $\widetilde{C}$

(ii) *the Macdonald's formula in type* $\widetilde{B}$

$$\eta(x)^{2t^2+t} = c_1 \sum \prod_i v_i \prod_{i<j} (v_i^2 - v_j^2) x^{\|v\|^2/(8(2t-1))},$$

where the sum is over t-uplets $(v_1, \ldots, v_t) \in \mathbb{Z}^t$ *such that* $v_i \equiv 2i - 1 \bmod 4t - 2$ *and* $v_1 + \cdots + v_t = t^2 \bmod 8t - 4$.

(iii) *the Macdonald's formula in type* $\widetilde{BC}$

$$\eta(x)^{2t^2-t} = c_2 \sum (-1)^{(v_1+\cdots+v_t-t)/2} \prod_{i<j} (v_i^2 - v_j^2) x^{\|v\|^2/(8(2t+1))},$$

where the sum ranges over t-uplets $(v_1, \ldots, v_t) \in \mathbb{Z}^t$ *such that* $v_i \equiv 2i - 1 \bmod 4t + 2$.

A number theoretic result

Write

$$\prod_{n\geq 1}(1-x^n)^s = \sum_{k\geq 0} f_k(s)x^k$$

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Theorem (P., 2014)

Let k be a positive integer and s be a real number such that $s > k-1$. Then $(-1)^k f_k(2s^2+s) > 0$.

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A generalization of Nekrasov-Okounkov formula in type $\tilde{C}$

Theorem (P., 2015)

Let $t = 2t' + 1$ be an odd positive integer. For any complex numbers y and z we have

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{yt(2z+2)}{\varepsilon_h \, h} \right) = \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1} \left(1 - x^{tk} y^{2k} \right)^{(2z+1)(zt+3t')}$$

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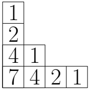

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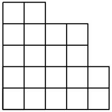

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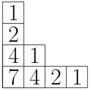

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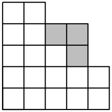

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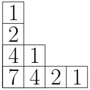

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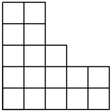

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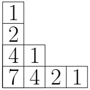

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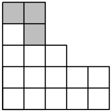

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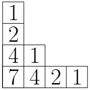

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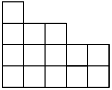

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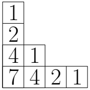

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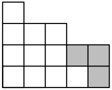

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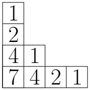

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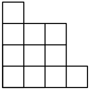

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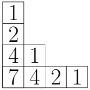

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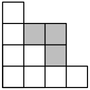

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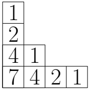

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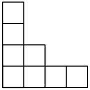

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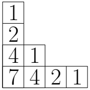

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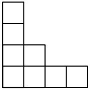

Fact : the t-core of a partition is a t-core.

Littlewood decomposition

Theorem (Littlewood, 1951, probably)

The *Littlewood decomposition* *maps a partition* λ *to* $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ *such that:*

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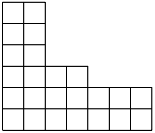

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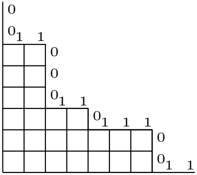

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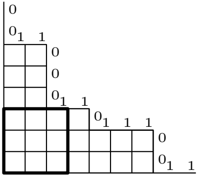

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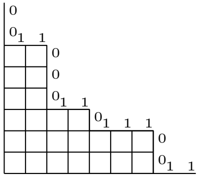

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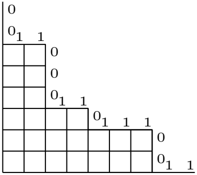

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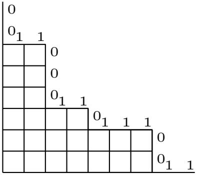

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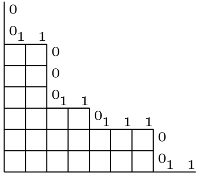

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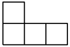
$\lambda^0 =$ 

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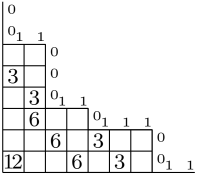

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New properties of Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

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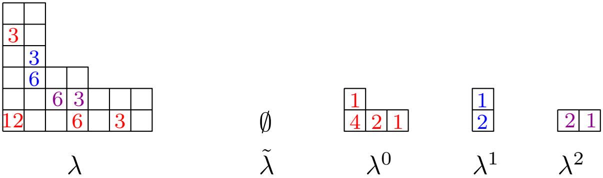

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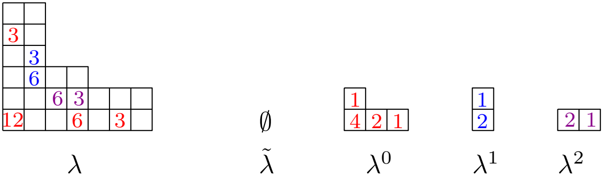

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(iv) two properties about the relative position of the boxes

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- And sum over all doubled distinct partitions.

Some consequences

Corollary (P., 2015)

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We have:

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{bt}{h\,\varepsilon_h} = \exp(-tb^2 x^t/2) \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1}$$

A new hook formula

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We have:

$$\sum_{\substack{\lambda \in DD,\ |\lambda|=2tn \\ \#\mathcal{H}_t(\lambda)=2n}} \delta_\lambda \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h\,\varepsilon_h} = \frac{(-1)^n}{n!\,t^n 2^n}$$

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Question: can we prove this by using the RSK algorithm?

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- Is there a generalization for t even? Involves $\widetilde{C}^{\vee}$
- What is the link with representation theory?
- What about other affine types (as $\widetilde{D}$)?

Thank you for your attention!